Variational Iteration Method for Delay Differential Equations Using He's Polynomials

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In this paper, we apply the variational iteration method using He's polynomials (VIMHP) for solving delay differential equations which are otherwise too difficult to solve. These equations arise very frequently in signal processing, digital images, physics, and applied sciences. Numerical results reveal the complete reliability and efficiency of the proposed combination.

Key words: Variational Iteration Method; He's Polynomials; Delay Differential Equations; Exact Solutions.

1. Introduction

Recently, He [1 – 14] developed the variational iteration (VIM) and the homotopy perturbation method (HPM) for solving a wide range of nonlinear problems. The subsequent works [15 – 36] clearly reflect the reliability and efficiency of these algorithms. Ghorbani et al. [23, 24] introduced He's polynomials which are compatible with Adomian's polynomials but are much easier to calculate. In a later work Noor and Mohyud-Din [29, 31 – 33] made the elegant coupling of He's polynomials and correction functional of VIM. The basic motivation of this paper is the extension of this very reliable combination (VIMHP) for solving linear and nonlinear delay differential equations (DDES) which are otherwise too difficult to handle because of their complex nature. These equations (DDES) [37] arise very frequently in signal processing, digital images, physics, and applied sciences. Moreover, delay differential equations (DDES) [38] are a large and important class of dynamical systems and have a wide range of applications in science and engineering. They arise when the rate of change of a time-dependant process in its mathematical modelling is not only determined by its present state but also by a certain past state. Recent studies in diverse fields such as biology, economics, control, and electromagnetic have shown that DDEs play an important role in explaining many physical phenomena. In particular, they turn out to be fundamental when ordinary differential equation (ODE) based models fail. The beauty of the variational iteration method using He's polynomials (VIMHP) is its accuracy, simplicity, and compatibility with the physical nature of the problems. The VIMHP is applied without any discretization, transformation or restrictive assumptions. The suggested method is free from round off errors and calculation of the so-called Adomian's polynomials. The proposed algorithm has been tested on various linear and nonlinear DDES of first, second and third-order. Numerical results are very encouraging and reveal the complete reliability of the proposed VIMHP.

2. Variational Iteration Method Using He's Polynomials (VIMHP)

To illustrate the basic concept of the VIMHP, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is the forcing term. According to VIM [6 – 22, 25, 28, 30 – 35], we can construct a so-called correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\mathrm{d}\xi, \tag{2}$$

where λ is a Lagrange multiplier [6 – 14], $\tilde{u}_n$ is a restricted variation. Now, we apply He's polynomials [23, 24]

$$\sum_{n=0}^{\infty} p^{(n)}u_n = u_0(x) + p\int_0^x \lambda(\xi)\Bigg(\sum_{n=0}^{\infty} p^{(n)}L(u_n) + \sum_{n=0}^{\infty} p^{(n)}N(\tilde{u}_n)\Bigg)\mathrm{d}\xi - \int_0^x \lambda(\xi)g(\xi)\mathrm{d}\xi, \tag{3}$$

which is the VIMHP [28, 30 – 32] and is formulated by the coupling of VIM and He's polynomials. The comparison of like powers of p gives solutions of various orders.

3. Numerical Applications

In this section, we apply the variational iteration method using He's polynomials (VIMHP) for solving linear and nonlinear DDES of various orders.

Example 3.1 Consider the following linear delay differential equation (LDDE) of second order:

$$\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2} = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 + 2, \quad 0 \le x \le 1,$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

The exact solution of the problem is

$$y(x) = x^2.$$

The correction functional is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[\frac{\mathrm{d}^2y_n(x)}{\mathrm{d}s^2} - \frac{3}{4}\tilde{y}_n(x) - \tilde{y}_n\left(\frac{x}{2}\right) + x^2 - 2\right]\mathrm{d}s.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s - x)$, we get

$$y_{n+1}(x) = y_n(x) + \int_0^x (s-x)\left[\frac{\mathrm{d}^2y_n(x)}{\mathrm{d}s^2} - \frac{3}{4}y_n(x) - y_n\left(\frac{x}{2}\right) + x^2 - 2\right]\mathrm{d}s.$$

Applying the variational iteration method using He's polynomials (VIMHP), we have

$$y_0 + py_1 + \cdots = y_0(x) + \int_0^x (s-x)\left[\left(\frac{\mathrm{d}^2y_0}{\mathrm{d}s^2} + p\frac{\mathrm{d}^2y_1}{\mathrm{d}s^2} + \cdots\right) - \frac{3}{4}(y_0 + py_1 + \cdots) - \left(y_0\left(\frac{x}{2}\right) + py_1\left(\frac{x}{2}\right) + \cdots\right) + x^2 - 2\right]\mathrm{d}s.$$

Comparing the co-efficient of like powers of p, following approximants are calculated:

$$p^{(0)}: y_0(x) = 0,$$
$$p^{(1)}: y_1(x) = x^2 - \frac{1}{12}x^4,$$
$$p^{(2)}: y_2(x) = \frac{13}{12}x^4 - \frac{13}{15360}x^6,$$
$$p^{(3)}: y_3(x) = \frac{13}{15360}x^6 - \frac{559}{18350080}x^8,$$
$$\vdots$$

The solution is given as

$$y(x) = x^2.$$

Example 3.2 Consider the following linear delay differential equation (LDDE) of first order:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{1}{2}\mathrm{e}^{x/2}y\left(\frac{x}{2}\right) + \frac{1}{2}y(x), \quad 0 \le x \le 1,$$

with the initial condition

$$y(0) = 1.$$

The exact solution of the problem is

$$y(x) = \mathrm{e}^x.$$

The correction functional is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[\frac{\mathrm{d}y_n(x)}{\mathrm{d}s} - \left(\frac{1}{2}\mathrm{e}^{x/2}\tilde{y}_n\left(\frac{x}{2}\right) + \frac{1}{2}\tilde{y}_n(x)\right)\right]\mathrm{d}s.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$, we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(\frac{\mathrm{d}y_n(x)}{\mathrm{d}s} - \left(\frac{1}{2}\mathrm{e}^{x/2}y_n\left(\frac{x}{2}\right) + \frac{1}{2}y_n(x)\right)\right)\mathrm{d}s.$$

Applying the variational iteration method using He's polynomials (VIMHP), we have

$$y_0 + py_1 + \cdots = y_0(x) - \int_0^x \left[\frac{\mathrm{d}y_0}{\mathrm{d}s} + p\frac{\mathrm{d}y_1}{\mathrm{d}s} + \cdots - \frac{1}{2}\mathrm{e}^{x/2}\left(y_0\left(\frac{x}{2}\right) + py_1\left(\frac{x}{2}\right) + \cdots\right) + \frac{1}{2}(y_0 + py_1 + \cdots)\right]\mathrm{d}s.$$

Comparing the co-efficient of like powers of p, following approximants are calculated:

$$p^{(0)}: y_0(x) = 1,$$
$$p^{(1)}: y_1(x) = x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \cdots,$$
$$p^{(2)}: y_2(x) = \frac{3}{8}x^2 + \frac{13}{192}x^3 + \frac{13}{1024}x^4 + \cdots,$$
$$p^{(3)}: y_3(x) = \frac{5}{64}x^3 + \frac{63}{4096}x^4 + \cdots,$$
$$\vdots$$

The series solution is given as

$$y(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots,$$

and the closed form solution is given by

$$y(x) = \mathrm{e}^x.$$

Example 3.3 Consider the following nonlinear delay differential equation (NDDE) of first order:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = 1 - 2y^2\left(\frac{x}{2}\right), \quad 0 \le x \le 1,$$

with the initial condition

$$y(0) = 0.$$

The exact solution of the problem is

$$y(x) = \sin x.$$

The correction functional is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[\frac{\mathrm{d}y_n(x)}{\mathrm{d}s} - 1 + 2\tilde{y}_n^2\left(\frac{x}{2}\right)\right]\mathrm{d}s.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$, we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[\frac{\mathrm{d}y_n(x)}{\mathrm{d}s} - 1 + 2y_n^2\left(\frac{x}{2}\right)\right]\mathrm{d}s.$$

Applying the variational iteration method using He's polynomials (VIMHP), we have

$$y_0 + py_1 + \cdots = y_0(x) - \int_0^x \left[\frac{\mathrm{d}y_0}{\mathrm{d}s} + p\frac{\mathrm{d}y_1}{\mathrm{d}s} + \cdots \right. \left. -1 + 2\left(y_0\left(\frac{x}{2}\right) + py_1\left(\frac{x}{2}\right) + \cdots\right)^2\right]\mathrm{d}s.$$

Comparing the co-efficient of like powers of p, following approximants are calculated:

$$p^{(0)}: y_0(x) = 0,$$
$$p^{(1)}: y_1(x) = x,$$
$$p^{(2)}: y_2(x) = 0,$$
$$p^{(3)}: y_3(x) = -\frac{1}{3!}x^3,$$
$$p^{(4)}: y_4(x) = 0,$$
$$p^{(5)}: y_5(x) = \frac{1}{5!}x^5,$$
$$p^{(6)}: y_6(x) = 0,$$
$$p^{(7)}: y_7(x) = -\frac{1}{7!}x^7,$$
$$p^{(8)}: y_8(x) = 0,$$
$$p^{(9)}: y_9(x) = \frac{1}{9!}x^9,$$
$$\vdots$$

The series solution is given as

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots,$$

and the closed form solution is given by

$$y(x) = \sin x.$$

Example 3.4 Consider the following nonlinear delay differential equation (NDDE) of third order:

$$\frac{\mathrm{d}^3y(x)}{\mathrm{d}x^3} = -1 + 2y^2\left(\frac{x}{2}\right), \quad 0 \le x \le 1,$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0.$$

The exact solution of the problem is

$$y(x) = \sin x.$$

The correction functional is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \left[\frac{\mathrm{d}^3y_n(x)}{\mathrm{d}s^3} + 1 - 2\tilde{y}_n^2\left(\frac{x}{2}\right)\right]\mathrm{d}s.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -\frac{1}{2!}(s-x)^2$, we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{1}{2!}(s-x)^2\left[\frac{\mathrm{d}^3y_n(x)}{\mathrm{d}s^3} + 1 - 2y_n^2\left(\frac{x}{2}\right)\right]\mathrm{d}s.$$

Applying the variational iteration method using He's polynomials (VIMHP), we have

$$y_0 + py_1 + \cdots =$$

$$y_0(x) - \int_0^x \frac{1}{2!}(s-x)^2 \left[\frac{d^3 y_0}{ds^3} + p\frac{d^3 y_1}{ds^3} + \cdots + 1 - 2\left(y_0\left(\frac{x}{2}\right) + p y_1\left(\frac{x}{2}\right) + \cdots \right)^2 \right] ds.$$

Comparing the co-efficient of like powers of p, following approximants are calculated:

$$p^{(0)} : y_0(x) = x - \frac{1}{3!}x^3,$$
$$p^{(1)} : y_1(x) = \tfrac{1}{5!}x^5 - \tfrac{1}{2880}x^7 + \tfrac{1}{580608}x^9,$$
$$\vdots$$

The series solution is given as

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots,$$

and the closed form solution is given by

$$y(x) = \sin x.$$

4. Conclusion

In this paper, we applied VIMHP on various linear and nonlinear DDES of first, second and third order. The proposed algorithm is employed without using linearization, discretization or restrictive assumptions. The fact that the VIMHP solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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[1] J.H. He, Top. Methods Nonlinear Anal. **31**, 205 (2008).
[2] J. H. He, Appl. Math. Comput. **156**, 527 (2004).
[3] J. H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[4] J. H. He, Appl. Math. Comput. **151**, 287 (2004).
[5] J. H. He, Int. J. Nonlinear Mech. **35**, 115 (2000).
[6] J. H. He, Int. J. Mod. Phys. B **22**, 3487 (2008).
[7] J. H. He, Int. J. Mod. Phys. (20)**10**, 1144 (2006).
[8] J. H. He, J. Comput. Appl. Math. **207**, 3 (2007).
[9] J.H. He and X. Wu, Comput. Math. Appl. **54**, 881 (2007).
[10] J. H. He, Int. J. Nonlinear Mech. **34**, 699 (1999).
[11] J. H. He, Appl. Math. Comput. **114**, 115 (2000).
[12] J.H. He and X.H. Wu, Chaos, Solitons, and Fractals **29**, 108 (2006).
[13] J. H. He, Phys. Scr. **76**, 680 (2007).
[14] S. Momani and Z. Odibat, Chaos, Solitons, and Fractals **27**, 1119 (2006).
[15] S. Abbasbandy, J. Comput. Appl. Math. **207**, 59 (2007).
[16] S. Abbasbandy, Int. J. Numer. Methods Eng. **70**, 876 (2007).
[17] M.A. Abdou and A.A. Soliman, J. Comput. Appl. Math. **181**, 245 (2005).
[18] M.A. Abdou and A.A. Soliman, Phys. D **211**, 1 (2005).
[19] T.A. Abassy, M.A. El-Tawil, and H. El-Zoheiry, J. Comput. Appl. Math. **207**, 73 (2007).
[20] B. Batiha, M.S.M. Noorani, and I. Hashim, Comput. Math. Appl. **54**, 903 (2007).
[21] J. Biazar and H. Ghazvini, Comput. Math. Appl. **54**, 1047 (2007).
[22] D.D. Ganji, H. Tari, and M.B. Jooybari, Comput. Math. Appl. **54**, 1018 (2007).
[23] A. Ghorbani and J. S. Nadjfi, Int. J. Nonlinear Sci. Numer. Simul. **8**, 229 (2007).
[24] A. Ghorbani, Chaos, Solitons, and Fractals **39**, 1486 (2009).
[25] A. Golbabai and M. Javidi, Comput. Math. Appl. **54**, 1003 (2007).
[26] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Nonlinear Sci. Numer. Simul. **10**, 223 (2009).
[27] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Nonlinear Sci. Numer. Simul. **10**, 581 (2009).
[28] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J. Mod. Phys. B (in press).
[29] S.T. Mohyud-Din and M.A. Noor, Z. Naturforsch. **64a**, 157 (2009).
[30] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9**, 141 (2008).
[31] M. A. Noor and S. T. Mohyud-Din, Acta Appl. Math. (2008), DOI: 10.1007/s10440-008-9255-x.
[32] M. A. Noor and S. T. Mohyud-Din, Math. Prob. Eng. 2008 (2008), Article ID 696734, DOI: 10.1155/2008/696734.
[33] M. Rafei and H. Daniali, Comput. Math. Appl. **54**, 1079 (2007).
[34] N. H. Sweilam, J. Comput. Appl. Math. **207**, 64 (2007).
[35] M. Tatari and M. Dehghan, J. Comput. Appl. Math. **207**, 121 (2007).
[36] L. Xu, Comput. Math. Appl. **54**, 1067 (2007).
[37] H.J. Oberle and H. Pesch, Numer. Math. **37**, 235 (1981).
[38] H. Kocak and A. Yildirim, Commun. Numer. Methods Eng. **25**, 1084 (2009).